21
Circle

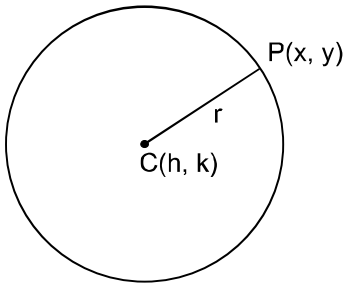
CIRCLE *A circle is the set of all points in a plane which are at a constant distance from a fixed point in the plane.*

The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

Equation of a Circle in Standard Form

THEOREM 1 *The equation of a circle with centre* (h, k) *and radius* r *is given by* $(x-h)^2+(y-k)^2=r^2$.

PROOF Let us consider a circle with centre $C(h, k)$ and radius r.



Let $P(x, y)$ be an arbitrary point on the circle.

Then, $|CP| = r \Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$

$\Rightarrow (x-h)^2+(y-k)^2 = r^2$,

which is the required equation of the given circle.

THEOREM 2 *The equation of a circle with centre at the origin and radius* r *is given by* $x^2+y^2=r^2$.

PROOF The equation of a circle with centre $C(0, 0)$ and radius r is given by

$(x-0)^2+(y-0)^2=r^2 \Rightarrow x^2+y^2=r^2$.

Equation of a Circle, the End Points of whose Diameter are given

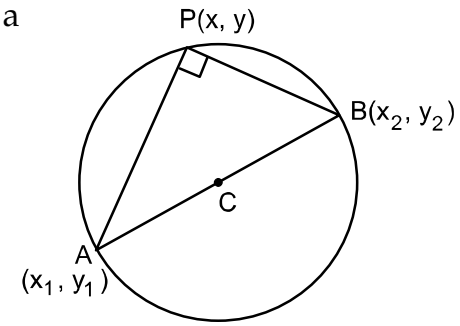
THEOREM 3 *The equation of a circle described on the line joining* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *as a diameter is* $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

PROOF Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of the given circle, and let $P(x, y)$ be any point on the circle.

Join AP and BP.

Then, the angle in a semicircle being a right angle,

we have $\angle APB = 90°$.



Now, slope of $AP = \dfrac{(y-y_1)}{(x-x_1)}$.

And, slope of $BP = \dfrac{(y-y_2)}{(x-x_2)}$.

Since $AP \perp BP$, we have

$$\frac{(y-y_1)}{(x-x_1)} \cdot \frac{(y-y_2)}{(x-x_2)} = -1 \Rightarrow (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0,$$

which is the required equation of the circle.

SUMMARY

(i) The equation of a circle with centre $C(h, k)$ and radius r is given by $\mathbf{(x-h)^2 + (y-k)^2 = r^2}$.

(ii) The equation of a circle with centre $C(0, 0)$ and radius r is given by $\mathbf{x^2 + y^2 = r^2}$.

(iii) The equation of a circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as the end points of a diameter is given by

$$\mathbf{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0.}$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of a circle with centre* $(3, -2)$ *and radius* 5.

SOLUTION We know that the equation of a circle with centre $C(h, k)$ and radius r is given by

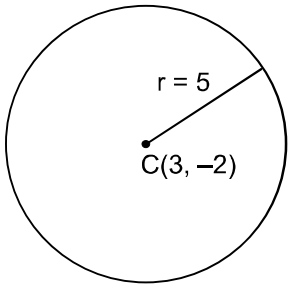
$$(x-h)^2 + (y-k)^2 = r^2.$$

Here, $h = 3$, $k = -2$ and $r = 5$.

$\therefore$ the required equation of the circle is

$$(x-3)^2 + (y+2)^2 = 5^2$$

$$\Rightarrow \quad x^2 + y^2 - 6x + 4y - 12 = 0.$$



EXAMPLE 2 *Find the equation of a circle whose centre is* $(2, -1)$ *and which passes through the point* $(3, 6)$.

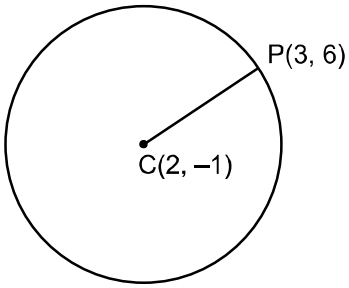
SOLUTION Let $C(2, -1)$ be the centre of the given circle and let it pass through the point $P(3, 6)$. Then, radius of the circle

$$= |CP| = \sqrt{(3-2)^2 + (6+1)^2} = \sqrt{50}.$$

$\therefore$ the required equation of the circle is

$$(x-2)^2 + (y+1)^2 = (\sqrt{50})^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 2y - 45 = 0.$$



EXAMPLE 3 *Find the equation of the circle passing through the point* $(2, 4)$ *and having its centre at the intersection of the lines* $x - y = 4$ *and* $2x + 3y + 7 = 0$.

SOLUTION The equations of the given lines are:

$$x - y = 4 \qquad \ldots \text{(i)}$$

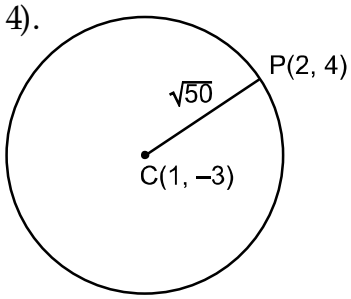
$$2x + 3y = -7 \qquad \ldots \text{(ii)}$$

Solving (i) and (ii) simultaneously, we get $x = 1$ and $y = -3$.

So, the point of intersection of the given lines is $C(1, -3)$.

$\therefore$ centre of the given circle is $C(1, -3)$.

Also, the circle passes through the point $P(2, 4)$.



$\therefore$ radius of the circle

$= |CP| = \sqrt{(1-2)^2 + (-3-4)^2} = \sqrt{50}$.

$\therefore$ the required equation of the circle is

$(x-1)^2 + (y+3)^2 = (\sqrt{50})^2$

$\Rightarrow \quad x^2 + y^2 - 2x + 6y - 40 = 0$.

EXAMPLE 4 *Find the equation of a circle of radius 5 units, whose centre lies on the x-axis and which passes through the point* $(2, 3)$.

SOLUTION It is given that the centre of the circle lies on the x-axis.

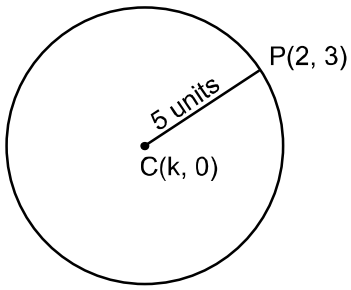
So, let $C(k, 0)$ be the centre of the circle.

Also, it is given that it passes through the point $P(2, 3)$.

$\therefore$ radius of the circle $= |CP|$

$= \sqrt{(k-2)^2 + (0-3)^2}$

$= \sqrt{k^2 - 4k + 13}$



$\Rightarrow \quad \sqrt{k^2 - 4k + 13} = 5$ [$\because$ radius $= 5$ units]

$\Rightarrow \quad k^2 - 4k + 13 = 25 \Rightarrow k^2 - 4k - 12 = 0$

$\Rightarrow \quad (k-6)(k+2) = 0 \Rightarrow k = 6$ or $k = -2$.

$\therefore$ centre of the circle is $(6, 0)$ or $(-2, 0)$.

Hence, the required equation of the circle is

$(x-6)^2 + (y-0)^2 = 5^2$ or $(x+2)^2 + (y-0)^2 = 5^2$

$\Rightarrow \quad x^2 + y^2 - 12x + 11 = 0$ or $x^2 + y^2 + 4x - 21 = 0$.

Thus, there are two circles satisfying the given conditions.

EXAMPLE 5 *Find the equation of a circle, the end points of one of whose diameters are* $A(2, -3)$ *and* $B(-3, 5)$.

SOLUTION We know that the equation of a circle, the end points of one of whose diameters are (x_1, y_1) and (x_2, y_2), is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

Here, $x_1 = 2$, $y_1 = -3$ and $x_2 = -3$, $y_2 = 5$.

$\therefore$ the required equation of the circle is

$(x-2)(x+3) + (y+3)(y-5) = 0$.

$\Rightarrow \quad x^2 + y^2 + x - 2y - 21 = 0$.

EXAMPLE 6 *Find the equation of the circle drawn on the diagonal of the rectangle as its diameter, whose sides are* $x = 6$, $x = -3$, $y = 3$ *and* $y = -1$.

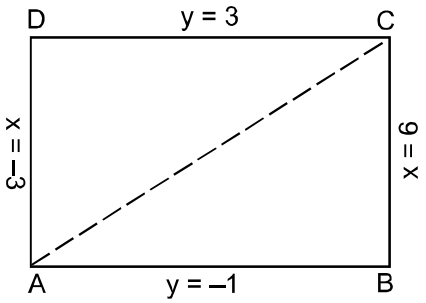
SOLUTION Let $ABCD$ be the given rectangle and let its sides AD, BC, AB and CD be represented by the equations $x = -3$, $x = 6$, $y = -1$ and $y = 3$ respectively.

Then, the coordinates of A and C are $A(-3, -1)$ and $C(6, 3)$.

So, the equation of the circle with AC as diameter is given by

$$(x + 3)(x - 6) + (y + 1)(y - 3) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 3x - 2y - 21 = 0.$$



EXAMPLE 7 *If $y = 2x$ is a chord of the circle $x^2 + y^2 - 10x = 0$, find the equation of the circle with this chord as a diameter.*

SOLUTION The points of intersection of the given chord and the given circle are obtained by simultaneously solving

$$y = 2x \text{ and } x^2 + y^2 - 10x = 0.$$

Putting $y = 2x$ in $x^2 + y^2 - 10x = 0$, we get

$$5x^2 - 10x = 0 \Leftrightarrow 5x(x - 2) = 0 \Leftrightarrow x = 0 \text{ or } x = 2.$$

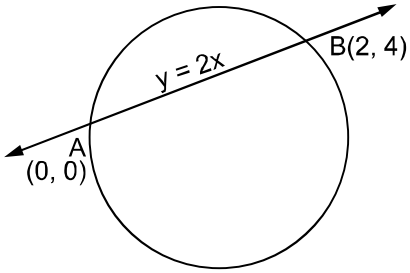
Now, $x = 0 \Rightarrow y = 0$, and $x = 2 \Rightarrow y = 4$.

$\therefore$ the points of intersection of the given chord and the given circle are $A(0, 0)$ and $B(2, 4)$.

$\therefore$ the required equation of the circle with AB as diameter is

$$(x - 0)(x - 2) + (y - 0)(y - 4) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 4y = 0.$$



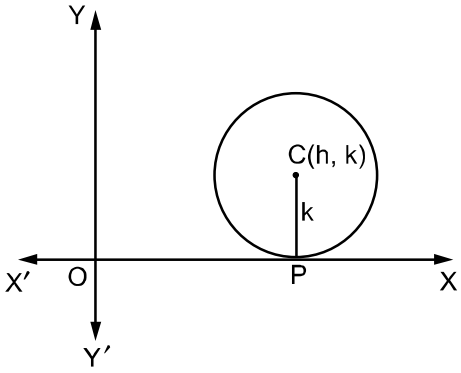
EXAMPLE 8 *Find the equation of a circle with centre (h, k) and touching the x-axis.*

SOLUTION Clearly, the radius of the circle $= k$.

So, the required equation is

$$(x - h)^2 + (y - k)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 = 0.$$



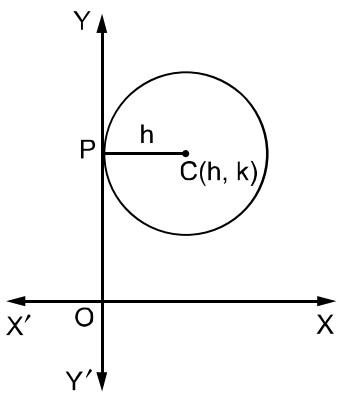
EXAMPLE 9 *Find the equation of a circle with centre (h, k) and touching the y-axis.*

SOLUTION Clearly, the radius of the circle $= h$.

So, the required equation is

$$(x - h)^2 + (y - k)^2 = h^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + k^2 = 0.$$



EXAMPLE 10 *Find the equation of a circle with centre* (h, k) *and touching both the axes.*

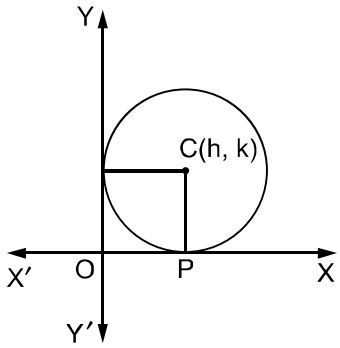
SOLUTION Clearly, radius $= h = k = c$ (say).

$\therefore$ the equation of the circle is

$(x - c)^2 + (y - c)^2 = c^2$

$\Rightarrow \quad x^2 + y^2 - 2c(x + y) + c^2 = 0,$

where $c = h = k$.



EXERCISE 21A

Find the equation of a circle with

1. centre (2, 4) and radius 5

2. centre (–3, –2) and radius 6

3. centre (a, a) and radius $\sqrt{2}$

4. centre $(a\cos\alpha, a\sin\alpha)$ and radius a

5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$

6. centre at the origin and radius 4

7. Find the centre and radius of each of the following circles:

(i) $(x - 3)^2 + (y - 1)^2 = 9$ (ii) $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{16}$

(iii) $(x + 5)^2 + (y - 3)^2 = 20$ (iv) $x^2 + (y - 1)^2 = 2$

8. Find the equation of the circle whose centre is $(2, -5)$ and which passes through the point (3, 2).

9. Find the equation of the circle of radius 5 cm, whose centre lies on the y-axis and which passes through the point (3, 2).

10. Find the equation of the circle whose centre is (2, –3) and which passes through the intersection of the lines $3x + 2y = 11$ and $2x + 3y = 4$.

11. Find the equation of the circle passing through the point (–1, 3) and having its centre at the point of intersection of the lines $x - 2y = 4$ and $2x + 5y + 1 = 0$.

12. If two diameters of a circle lie along the lines $x - y = 9$ and $x - 2y = 7$, and the area of the circle is 38.5 sq cm, find the equation of the circle.

13. Find the equation of the circle, the coordinates of the end points of one of whose diameters are

(i) $A(3, 2)$ and $B(2, 5)$ (ii) $A(5, -3)$ and $B(2, -4)$

(iii) $A(-2, -3)$ and $B(-3, 5)$ (iv) $A(p, q)$ and $B(r, s)$

14. The sides of a rectangle are given by the equations $x = -2$, $x = 4$, $y = -2$ and $y = 5$. Find the equation of the circle drawn on the diagonal of this rectangle as its diameter.

ANSWERS (EXERCISE 21A)

1. $x^2 + y^2 - 4x - 8y - 5 = 0$ **2.** $x^2 + y^2 + 6x + 4y - 23 = 0$

3. $x^2 + y^2 - 2ax - 2ay = 0$ **4.** $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$

5. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$ **6.** $x^2 + y^2 - 16 = 0$

7. (i) Centre (3, 1), radius = 3 (ii) Centre $\left(\frac{1}{2}, \frac{-1}{3}\right)$, radius = $\frac{1}{4}$

(iii) Centre (–5, 3), radius = $2\sqrt{5}$ (iv) Centre (0, 1), radius = $\sqrt{2}$

8. $x^2 + y^2 - 4x + 10y - 21 = 0$

9. $(x^2 + y^2 - 12y + 11 = 0)$ or $(x^2 + y^2 + 4y - 21 = 0)$

10. $x^2 + y^2 - 4x + 6y + 3 = 0$ **11.** $x^2 + y^2 - 4x + 2y - 20 = 0$

12. $4x^2 + 4y^2 - 88x - 16y + 451 = 0$

13. (i) $x^2 + y^2 - 5x - 7y + 16 = 0$ (ii) $x^2 + y^2 - 7x + 7y + 22 = 0$

(iii) $x^2 + y^2 + 5x - 2y - 9 = 0$ (iv) $(x - p)(x - r) + (y - q)(y - s) = 0$

14. $x^2 + y^2 - 2x - 3y - 18 = 0$

HINTS TO SOME SELECTED QUESTIONS

9. Let the centre of the circle be $C(0, k)$.

It is given that the circle passes through $P(3, 2)$.

Then, $|CP|^2 = 5^2 \Leftrightarrow (0 - 3)^2 + (2 - k)^2 = 5^2$

$\Leftrightarrow (2 - k) = \pm 4 \Leftrightarrow k = 6$ or $k = -2$.

$\therefore$ the centre of the circle is $(0, 6)$ or $(0, -2)$.

Hence, the equation of the circle is

$[(x - 0)^2 + (y - 6)^2 = 5^2]$ or $[(x - 0)^2 + (y + 2)^2 = 5^2]$.

12. We know that the point of intersection of two diameters of a circle is the centre of the circle.

On solving $x - y = 9$ and $x - 2y = 7$, we get the centre as $C(11, 2)$.

Now, $\pi r^2 = \frac{77}{2} \Leftrightarrow |r^2| = \left(\frac{77}{2} \times \frac{7}{22}\right) = \frac{49}{4} \Leftrightarrow r = \frac{7}{2}$.

$\therefore$ the required equation is $(x - 11)^2 + (y - 2)^2 = \frac{49}{4}$.

GENERAL EQUATION OF A CIRCLE

THEOREM *The general equation of a circle is of the form* $x^2 + y^2 + 2gx + 2fy + c = 0$. *Also, every equation of the form* $x^2 + y^2 + 2gx + 2fy + c = 0$ *represents a circle if* $(g^2 + f^2 - c) > 0$.

PROOF We know that the equation of a circle with centre (h, k) and radius r is given by

$(x - h)^2 + (y - k)^2 = r^2$

$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + c = 0$, where $-h = g$, $-k = f$ and $(h^2 + k^2 - r^2) = c$.

Hence, the general equation of a circle is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$.

Conversely, let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the given equation. Then,

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow \quad (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = (g^2 + f^2 - c)$

$\Rightarrow \quad (x + g)^2 + (y + f)^2 = (\sqrt{g^2 + f^2 - c})^2$

$\Rightarrow \quad \{x - (-g)\}^2 + \{y - (-f)\}^2 = (\sqrt{g^2 + f^2 - c})^2$

$\Rightarrow \quad (x - h)^2 + (y - k)^2 = r^2$, where $h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$.

Clearly, this equation will represent a circle if $(g^2 + f^2 - c) > 0$.

The centre of this circle is $(-g, -f)$ and its radius is $\sqrt{g^2 + f^2 - c}$.

NOTE (i) If $(g^2 + f^2 - c) = 0$ then the above equation represents a *point circle.*

(ii) If $(g^2 + f^2 - c) < 0$ then the above equation does not represent a circle.

REMARK The equation of a circle has the following properties:

(i) It is a second-degree equation in x and y.

(ii) It contains no term of xy.

(iii) Coefficient of x^2 = coefficient of y^2.

Thus, the equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ will represent a circle only when $a = b$ and $h = 0$.

SUMMARY

The general equation of a circle is

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0.}$$

Its centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the equation* $x^2 + y^2 - 6x + 4y - 36 = 0$ *represents a circle. Also, find its centre and radius.*

SOLUTION The given equation is $x^2 + y^2 - 6x + 4y - 36 = 0$.

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$,

where $2g = -6$, $2f = 4$ and $c = -36$.

$\therefore \quad g = -3$, $f = 2$ and $c = -36$.

Hence, the given equation represents a circle.

Centre of the circle $= (-g, -f) = (3, -2)$.

Radius of the circle $= \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + (-2)^2 + 36} = \sqrt{49} = 7$ units.

EXAMPLE 2 *Show that the equation* $3x^2 + 3y^2 + 12x - 18y - 11 = 0$ *represents a circle. Also, find its centre and radius.*

SOLUTION $3x^2 + 3y^2 + 12x - 18y - 11 = 0$

$$\Rightarrow \quad x^2 + y^2 + 4x - 6y - \frac{11}{3} = 0. \qquad \ldots (i)$$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$,

where $2g = 4$, $2f = -6$ and $c = -\frac{11}{3}$.

$\therefore \quad g = 2$, $f = -3$ and $c = -\frac{11}{3}$.

Hence, the given equation represents a circle.

Centre of the circle $= (-g, -f) = (-2, 3)$.

Radius of the circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(-2)^2 + 3^2 + \frac{11}{3}} = \sqrt{\frac{50}{3}} \text{ units.}$$

EXAMPLE 3 *Find the equation of a circle passing through the points* $(5, 7)$, $(6, 6)$ *and* $(2, -2)$. *Find its centre and radius.*

SOLUTION Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad \ldots (i)$$

Since it passes through each of the points $(5, 7)$, $(6, 6)$ and $(2, -2)$, each one of these points must satisfy (i).

$\therefore \quad 25 + 49 + 10g + 14f + c = 0 \Rightarrow 10g + 14f + c + 74 = 0 \qquad \ldots (ii)$

$36 + 36 + 12g + 12f + c = 0 \Rightarrow 12g + 12f + c + 72 = 0 \qquad \ldots (iii)$

$4 + 4 + 4g - 4f + c = 0 \Rightarrow 4g - 4f + c + 8 = 0 \qquad \ldots (iv)$

Subtracting (ii) from (iii), we get

$$2g - 2f - 2 = 0 \Rightarrow g - f = 1. \qquad \ldots (v)$$

Subtracting (iv) from (iii), we get

$$8g + 16f + 64 = 0 \Rightarrow g + 2f = -8. \qquad \ldots (vi)$$

Solving (v) and (vi), we get $g = -2$ and $f = -3$.

Putting $g = -2$ and $f = -3$ in (ii), we get $c = -12$.

Putting $g = -2$, $f = -3$ and $c = -12$ in (i), we get

$$x^2 + y^2 - 4x - 6y - 12 = 0,$$

which is the required equation of the circle.

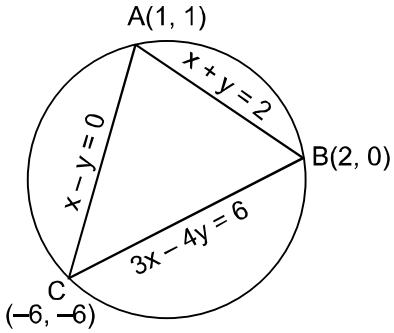
Centre of this circle $= (-g, -f) = (2, 3)$.

And, its radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 + 12} = \sqrt{25} = 5$ units.

EXAMPLE 4 *Find the equation of the circle passing through the vertices of a triangle whose sides are represented by the equations* $x + y = 2$, $3x - 4y = 6$ *and* $x - y = 0$.

SOLUTION Let the sides AB, BC and CA of $\triangle ABC$ be represented by the equations $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$ respectively.

Solving $x + y = 2$ and $3x - 4y = 6$, we get $B(2, 0)$.
Solving $3x - 4y = 6$ and $x - y = 0$, we get $C(-6, -6)$.
Solving $x + y = 2$ and $x - y = 0$, we get $A(1, 1)$.



Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \text{(i)}$$

Since it passes through $A(1, 1)$, $B(2, 0)$ and $C(-6, -6)$, each of these points must satisfy (i).

$\therefore \quad 1^2 + 1^2 + 2g + 2f + c = 0 \Rightarrow 2g + 2f + c + 2 = 0 \quad \ldots \text{(ii)}$

$2^2 + 0^2 + 4g + c = 0 \Rightarrow 4g + c + 4 = 0 \quad \ldots \text{(iii)}$

$(-6)^2 + (-6)^2 - 12g - 12f + c = 0 \Rightarrow -12g - 12f + c + 72 = 0 \quad \ldots \text{(iv)}$

Subtracting (ii) from (iii), we get

$$2g - 2f + 2 = 0 \Leftrightarrow g - f = -1. \quad \ldots \text{(v)}$$

Subtracting (iv) from (iii), we get

$$16g + 12f - 68 = 0 \Leftrightarrow 4g + 3f = 17. \quad \ldots \text{(vi)}$$

Solving (v) and (vi), we get $g = 2$ and $f = 3$.
Putting $g = 2$ in (iii), we get $c = -12$.
Hence, the required equation is

$$x^2 + y^2 + 4x + 6y - 12 = 0.$$

EXAMPLE 5 *Show that the points* $(5, 5)$, $(6, 4)$, $(-2, 4)$ *and* $(7, 1)$ *are concyclic, i.e., all lie on the same circle. Find the equation, centre and radius of this circle.*

SOLUTION Let the equation of the circle passing through the points (5, 5), (6, 4) and (7, 1) be given by

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \text{(i)}$$

Then, each of these points must satisfy (i).

$25 + 25 + 10g + 10f + c = 0 \Rightarrow 10g + 10f + c = -50. \quad \ldots \text{(ii)}$

$36 + 16 + 12g + 8f + c = 0 \Rightarrow 12g + 8f + c = -52. \quad \ldots \text{(iii)}$

$49 + 1 + 14g + 2f + c = 0 \Rightarrow 14g + 2f + c = -50. \quad \ldots \text{(iv)}$

Subtracting (ii) from (iv), we get

$$4g - 8f = 0 \Rightarrow g - 2f = 0. \quad \ldots \text{(v)}$$

Subtracting (iii) from (iv), we get

$$2g - 6f = 2 \Rightarrow g - 3f = 1. \quad \ldots \text{(vi)}$$

Solving (v) and (vi), we get $g = -2$ and $f = -1$.
Putting these values in (ii), we get $c = -20$.
Putting $g = -2$, $f = -1$ and $c = -20$ in (i), we get

$$x^2 + y^2 - 4x - 2y - 20 = 0. \quad \ldots \text{(vii)}$$

This is the equation of the circle passing through the points (5, 5), (6, 4) and (7, 1).

Putting $x = -2$ and $y = 4$ in (vii), we get

LHS $= 4 + 16 + 8 - 8 - 20 = 0 =$ RHS.

This shows that the point $(-2, 4)$ also lies on (vii).

Hence, the points $(5, 5)$, $(6, 4)$, $(-2, 4)$ and $(7, 1)$ all lie on the same circle, given by (vii).

Centre of this circle $= (-g, -f) = (2, 1)$.

Radius of this circle $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-1)^2 + 20}$

$= \sqrt{25} = 5$ units.

EXAMPLE 6 *Find the equation of the circle which passes through the centre of the circle* $x^2 + y^2 + 8x + 10y - 7 = 0$ *and is concentric with the circle* $2x^2 + 2y^2 - 8x - 12y - 9 = 0$.

SOLUTION We have $2x^2 + 2y^2 - 8x - 12y - 9 = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y - \frac{9}{2} = 0 \Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0,$

where $g = -2$, $f = -3$ and $c = \frac{-9}{2}$.

Centre of this circle $= (-g, -f) = (2, 3)$.

$\therefore$ the centre of the required circle $= C(2, 3)$.

Again, $x^2 + y^2 + 8x + 10y - 7 = 0$

$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + c = 0$, where $g = 4$, $f = 5$ and $c = -7$.

Centre of this circle $= (-g, -f) = (-4, -5)$.

So, the required circle passes through the point $P(-4, -5)$.

Radius of the required circle $= CP = \sqrt{(2 + 4)^2 + (3 + 5)^2}$

$= \sqrt{36 + 64} = \sqrt{100} = 10.$

Hence, the required equation of the circle is

$(x - 2)^2 + (y - 3)^2 = (10)^2 \Rightarrow x^2 + y^2 - 4x - 6y - 87 = 0.$

EXAMPLE 7 *Find the equation of the circle whose centre lies on the line* $x - 4y = 1$ *and which passes through the points* $(3, 7)$ *and* $(5, 5)$.

SOLUTION Let the required equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0.$ ... (i)

Its centre is $(-g, -f)$, which lies on the line $x - 4y = 1$.

$\therefore \quad -g + 4f = 1.$ ... (ii)

Also, it is given that (i) passes through the points (3, 7) and (5, 5).

$\therefore \quad 3^2 + 7^2 + 6g + 14f + c = 0 \Rightarrow 6g + 14f + c = -58.$ ... (iii)

$5^2 + 5^2 + 10g + 10f + c = 0 \Rightarrow 10g + 10f + c = -50.$ ... (iv)

Subtracting (iii) from (iv), we get

$4g - 4f = 8 \Leftrightarrow g - f = 2.$ ... (v)

Solving (ii) and (v), we get $g = 3$ and $f = 1$.

Putting $g = 3$ and $f = 1$ in (iii), we get $c = -90$.

$\therefore \quad g = 3,\ f = 1$ and $c = -90$.

Putting these values in (i), we get the required equation as

$$x^2 + y^2 + 6x + 2y - 90 = 0.$$

EXAMPLE 8 *Find the equation of a circle concentric with the circle*

$$2x^2 + 2y^2 - 6x + 8y + 1 = 0$$

and of double its area.

SOLUTION The given equation of the circle is

$$2x^2 + 2y^2 - 6x + 8y + 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 3x + 4y + \frac{1}{2} = 0 \Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0,$$

where $g = \frac{-3}{2}$, $f = 2$ and $c = \frac{1}{2}$.

Centre of this circle $= (-g, -f) = \left(\frac{3}{2}, -2\right)$.

Radius of this circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\frac{9}{4} + 4 - \frac{1}{2}} = \frac{\sqrt{23}}{2}.$$

$\therefore$ the centre of the required circle $= \left(\frac{3}{2}, -2\right)$.

Let r_1 be the radius of the required circle. Then,

$$\pi r_1^2 = 2\left\{\pi \times \left(\frac{\sqrt{23}}{2}\right)^2\right\} \Rightarrow r_1^2 = \frac{23}{2}.$$

Hence, the equation of the required circle is

$$\left(x - \frac{3}{2}\right)^2 + (y + 2)^2 = \frac{23}{2}$$

$$\Rightarrow \quad x^2 + y^2 - 3x + 4y - \frac{21}{4} = 0$$

$$\Rightarrow \quad 4x^2 + 4y^2 - 12x + 16y - 21 = 0.$$

EXAMPLE 9 *Find the equation of a circle which passes through the origin and cuts off intercepts –2 and 3 from the x-axis and the y-axis respectively.*

SOLUTION Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad \ldots \text{(i)}$$

Clearly, the circle passes through the points $O(0, 0)$, $A(-2, 0)$ and $B(0, 3)$.

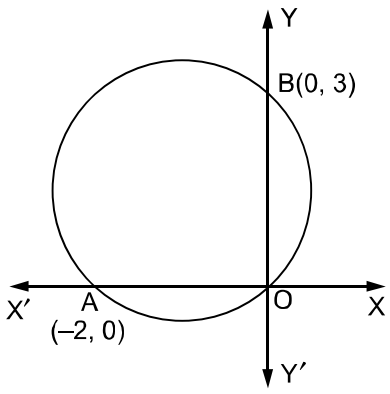
Putting $x = 0$ and $y = 0$ in (i), we get $c = 0$.

Thus, (i) becomes

$$x^2 + y^2 + 2gx + 2fy = 0. \qquad \ldots \text{(ii)}$$

Putting $x = -2$ and $y = 0$ in (ii), we get

$$4g = 4 \Leftrightarrow g = 1.$$



Putting $x = 0$ and $y = 3$ in (ii), we get

$$6f = -9 \Leftrightarrow f = \frac{-3}{2}.$$

Putting $g = 1$ and $f = \frac{-3}{2}$ in (ii), we get $x^2 + y^2 + 2x - 3y = 0$,

which is the required equation of the circle.

EXERCISE 21B

1. Show that the equation $x^2 + y^2 - 4x + 6y - 5 = 0$ represents a circle. Find its centre and radius.

2. Show that the equation $x^2 + y^2 + x - y = 0$ represents a circle. Find its centre and radius.

3. Show that the equation $3x^2 + 3y^2 + 6x - 4y - 1 = 0$ represents a circle. Find its centre and radius.

4. Show that the equation $x^2 + y^2 + 2x + 10y + 26 = 0$ represents a point circle. Also, find its centre.

5. Show that the equation $x^2 + y^2 - 3x + 3y + 10 = 0$ does not represent a circle.

6. Find the equation of the circle passing through the points

(i) $(0, 0)$, $(5, 0)$ and $(3, 3)$

(ii) $(1, 2)$, $(3, -4)$ and $(5, -6)$

(iii) $(20, 3)$, $(19, 8)$ and $(2, -9)$

Also, find the centre and radius in each case.

7. Find the equation of the circle which is circumscribed about the triangle whose vertices are $A(-2, 3)$, $B(5, 2)$ and $C(6, -1)$. Find the centre and radius of this circle.

8. Find the equation of the circle concentric with the circle $x^2 + y^2 + 4x + 6y + 11 = 0$ and passing through the point $P(5, 4)$.

9. Show that the points $A(1, 0)$, $B(2, -7)$, $C(8, 1)$ and $D(9, -6)$ all lie on the same circle. Find the equation of this circle, its centre and radius.

10. Find the equation of the circle which passes through the points $(1, 3)$ and $(2, -1)$, and has its centre on the line $2x + y - 4 = 0$.

11. Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ and which touches the y-axis.

12. Find the equation of the circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and of double its area.

13. Prove that the centres of the three circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and $x^2 + y^2 - 10x - 16y + 7 = 0$ are collinear.

14. Find the equation of the circle which passes through the points $A(1, 1)$ and $B(2, 2)$ and whose radius is 1. Show that there are two such circles.

15. Find the equation of a circle passing through the origin and intercepting lengths a and b on the axes.

16. Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.

17. Show that the quadrilateral formed by the straight lines $x - y = 0$, $3x + 2y = 5$, $x - y = 10$ and $2x + 3y = 0$ is cyclic and hence find the equation of the circle.

18. If $(-1, 3)$ and (α, β) are the extremities of the diameter of the circle $x^2 + y^2 - 6x + 5y - 7 = 0$, find the coordinates (α, β).

ANSWERS (EXERCISE 21B)

1. Centre $(2, -3)$, radius $= 3\sqrt{2}$

2. Centre $\left(-\frac{1}{2}, \frac{1}{2}\right)$, radius $= \frac{1}{\sqrt{2}}$

3. Centre $\left(-1, \frac{2}{3}\right)$, radius $= \frac{4}{3}$

4. Centre $(-1, -5)$

6. (i) $x^2 + y^2 - 5x - y = 0$, centre $\left(\frac{5}{2}, \frac{1}{2}\right)$ and radius $= \frac{\sqrt{26}}{2}$

(ii) $x^2 + y^2 - 22x - 4y + 25 = 0$, centre $(11, 2)$ and radius $= 10$

(iii) $x^2 + y^2 - 14x - 6y - 111 = 0$, centre $(7, 3)$ and radius $= 13$

7. $x^2 + y^2 - 2x + 2y - 23 = 0$, centre $(1, -1)$ and radius $= 5$

8. $x^2 + y^2 + 4x + 6y - 85 = 0$

9. $x^2 + y^2 - 10x + 6y + 9 = 0$, centre $C(5, -3)$ and radius $= 5$

10. $x^2 + y^2 - 3x - 2y - 1 = 0$

11. $x^2 + y^2 - 4x - 6y + 9 = 0$

12. $x^2 + y^2 - 6x + 12y - 15 = 0$

14. $x^2 + y^2 - 4x - 2y + 4 = 0$, $x^2 + y^2 - 2x - 4y + 4 = 0$

15. $x^2 + y^2 - ax - by = 0$

16. $x^2 + y^2 - 17x - 19y + 50 = 0$

17. $x^2 + y^2 - 6x + 4y = 0$

18. $B(7, -8)$

HINTS TO SOME SELECTED QUESTIONS

4. Show that the radius of the circle is 0.

5. Show that $(g^2 + f^2 - c) < 0$, where $g = \frac{-3}{2}$, $f = \frac{3}{2}$ and $c = 10$.

7. Find the equation of the circle passing through the points $A(-2, 3)$, $B(5, 2)$ and $C(6, -1)$.

8. Centre of the required circle is $C(-2, -3)$.

The circle passes through the point $P(5, 4)$.

Radius $= |CP| = \sqrt{(5 + 2)^2 + (4 + 3)^2} = 7\sqrt{2}$.

$\therefore$ the required equation is $(x + 2)^2 + (y + 3)^2 = 98$.

11. Centre of the given circle is $(2, 3)$.

Centre of the required circle is $(2, 3)$.

The required circle touches the y-axis, so its radius $= 2$.

$\therefore$ the required equation is $(x - 2)^2 + (y - 3)^2 = 2^2$.

13. Centres of the given circles are $A(2, 3)$, $B(-1, -2)$ and $C(5, 8)$ respectively. Now, show that $BA + AC = BC$.

Hence B, A, C are collinear.

14. Let the centre of the circle be $C(h, k)$.

Then, $|CA| = 1$ and $|CB| = 1$

$\Rightarrow \quad (h-1)^2 + (k-1)^2 = 1^2$ and $(h-2)^2 + (k-2)^2 = 1^2$

$\Rightarrow \quad h^2 + k^2 - 2h - 2k + 1 = 0$ … (i)

and $h^2 + k^2 - 4h - 4k + 7 = 0.$ … (ii)

Subtracting (ii) from (i), we get

$$2h + 2k - 6 = 0 \Rightarrow h + k = 3 \Rightarrow k = (3 - h).$$

Putting this value in (i), we get

$$h^2 + (3-h)^2 - 2h - 2(3-h) + 1 = 0$$

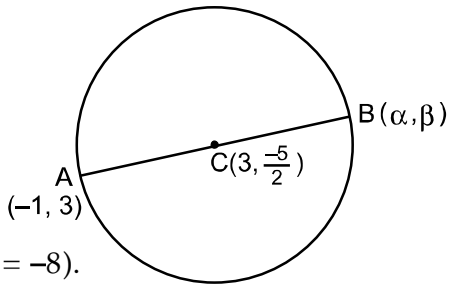
$\Rightarrow \quad (h-2)(h-1) = 0 \Rightarrow h = 2$ or $h = 1$.

$\therefore \quad k = 1$ or $k = 2$.

Hence, the centres are (2, 1) and (1, 2).

15. The circle passes through the points $O(0, 0)$, $A(a, 0)$ and $B(0, b)$.

18. Centre of the circle is $C\left(3, -\frac{5}{2}\right)$.



One end of the diameter AB is $A(-1, 3)$.

Let the other end of the diameter be $B(\alpha, \beta)$.

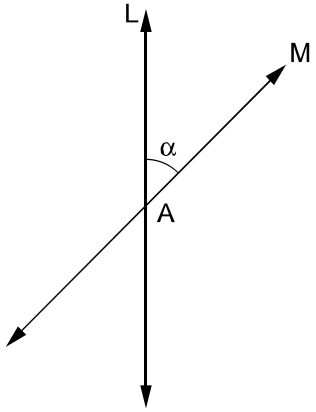
Then, C is the midpoint of AB.

$$\therefore \quad \left(\frac{-1+\alpha}{2} = 3 \text{ and } \frac{3+\beta}{2} = \frac{-5}{2}\right) \Rightarrow (\alpha = 7 \text{ and } \beta = -8).$$

22

Parabola

CONIC SECTIONS Under this heading, we shall study about the curves, namely *parabola, ellipse* and *hyperbola*. These curves can be obtained as intersections of a plane with a double-napped right circular cone.

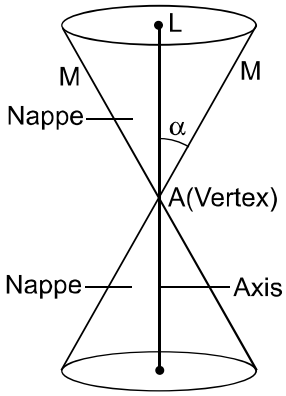


NAPPES Let L be a fixed vertical line and let M be another line intersecting it at a fixed point A and inclined to it at an angle α.

When M is rotated around the line L such that α remains constant, then the surface generated is a double-napped right circular hollow cone.

The line L is called the *axis* of the cone, A is called its *vertex* and the line M is called the *generator*.

The two parts, separated by the point A are called *nappes*.



CONIC SECTIONS *When a right circular cone is intersected by a plane, the curves obtained are known as conic sections.*

INTERSECTION OF NAPPES OF A CONE AND A PLANE

Suppose that a plane cuts double-napped cones each with semi-vertical angle α.

Let β be the angle made by the plane with the axis of the cone.

CASE 1 *When the plane cuts the nappes, but not at the vertex*

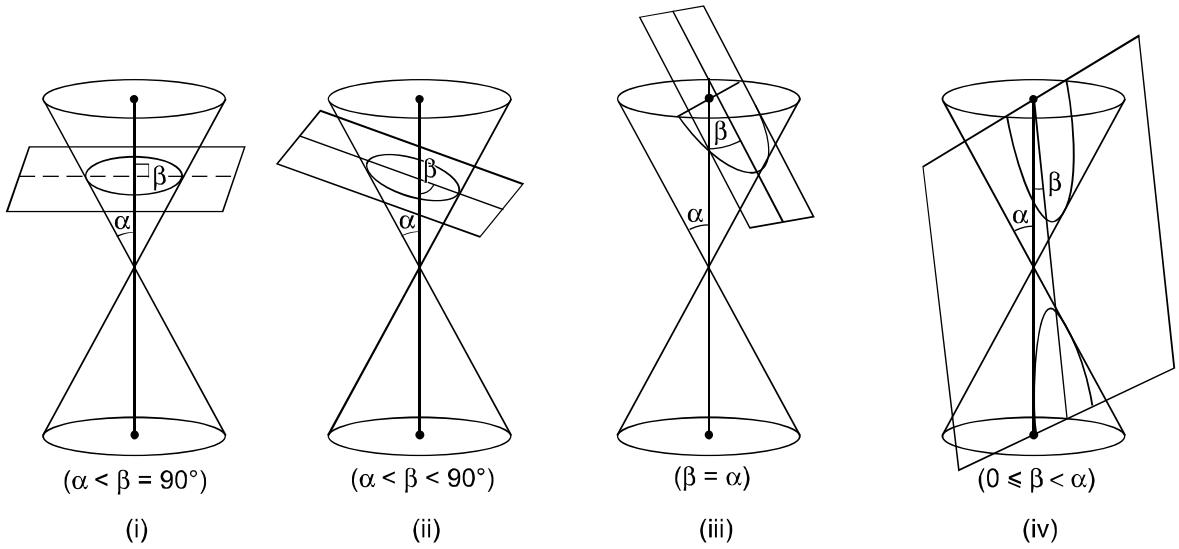
In this case, the following situations arise.

(i) When $\alpha < \beta = 90°$, the section is a ***circle***. [See Fig. (i)]

(ii) When $\alpha < \beta < 90°$, the section is an ***ellipse***. [See Fig. (ii)]

(iii) When $\beta = \alpha$, the section is a ***parabola***. [See Fig. (iii)]

(iv) When $0 \le \beta < \alpha$, the plane cuts both the nappes and the curves of intersection is a ***hyperbola***. [See Fig. (iv)]



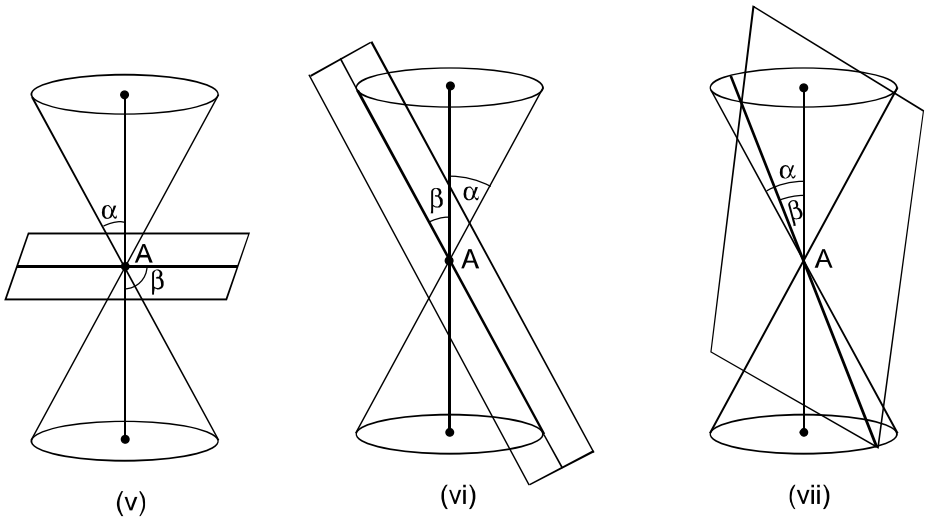
CASE 2 *When the plane cuts the nappe at the vertex*

In this case, the following situations arise.

(i) When $\alpha < \beta \le 90^\circ$, then the section is a ***point.*** [Fig. (v)]

(ii) When $\beta = \alpha$, then the section is a ***line.*** [Fig. (vi)]

(iii) When $0 \le \beta < \alpha$, then the section is a ***pair of straight lines.*** [Fig. (vii)]



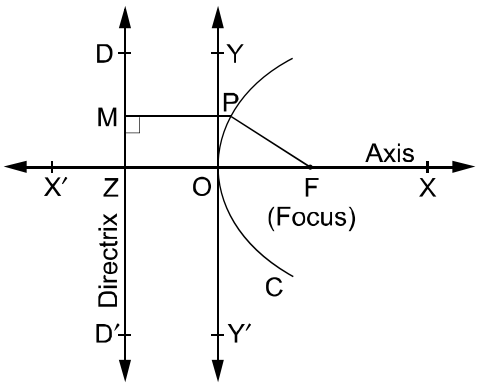
PARABOLA

PARABOLA *It is the path traced by a point which moves in a plane in such a way that its distance from a fixed point is always equal to its distance from a fixed line, both lying in the same plane, whereas the given fixed point does not lie on the given line.*

The fixed point is called the *focus* of the parabola and the fixed line is called its *directrix*.

A line through the focus and perpendicular to the directrix is called the *axis* of the parabola.

The point of intersection of the parabola with its axis is called the *vertex* of the parabola.



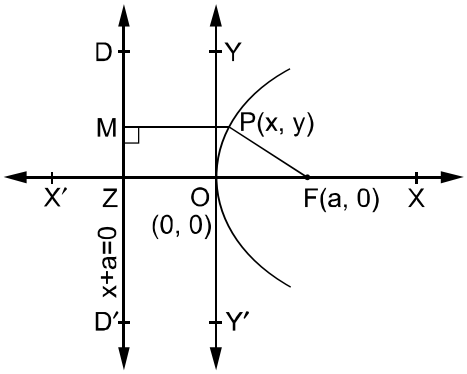
In the adjoining figure, C is a parabola with focus F and the line DD' as its directrix.

Clearly, x-axis is the axis of the parabola and $O(0, 0)$ is its vertex.

If we take an arbitrary point P on the parabola and draw $PM \perp DD'$ then by the definition of a parabola, we have $PF = PM$.

RIGHT-HANDED PARABOLA (or First Standard Form): $y^2 = 4ax,\ a > 0$

Let $X'OX$ and YOY' be the coordinate axes and let $a > 0$ be given.



Let us consider a parabola whose focus is $F(a, 0)$ and the directrix is the line DD', whose equation is $x + a = 0$.

Let $P(x, y)$ be an arbitrary point on the parabola. Let $PM \perp DD'$.

Then, by the definition of a parabola, we have $PF = PM$.

Now, $PF = PM \Rightarrow PF^2 = PM^2$

$\Rightarrow (x - a)^2 + y^2 = (x + a)^2$ [$\because$ length of perp. from $P(x, y)$ on the line $(x + a) = 0$ is $(x + a)$]

$\Rightarrow y^2 = (x + a)^2 - (x - a)^2$

$\Rightarrow y^2 = 4ax\ (a > 0)$.

Thus, every point on the parabola satisfies the equation, $y^2 = 4ax$.

Conversely, let $P(x, y)$ be a point satisfying the equation, $y^2 = 4ax$.

Let $F(a, 0)$ be a given point and $(x + a) = 0$ be a given line DD', and let $PM \perp DD'$. Then,

$$PF = \sqrt{(x - a)^2 + y^2}$$

$$= \sqrt{(x - a)^2 + 4ax} \qquad [\because\ y^2 = 4ax]$$

$$= \sqrt{(x + a)^2}$$

$$= (x + a) = PM$$

[$\because\ PM =$ length of perp. from $P(x, y)$ on the line $(x + a) = 0$].

This shows that $P(x, y)$ lies on the parabola, $y^2 = 4ax$.

Hence, the equation of a parabola with vertex at the origin, focus at the point $F(a, 0)$ and directrix $x = -a$ is $y^2 = 4ax$.

This equation is known as the *first standard form* of a parabola or a *right-handed parabola*.

FOCAL CHORD *Any chord of a parabola passing through its focus is called a focal chord.*

FOCAL DISTANCE OF A POINT *The distance of any point of a parabola from its focus is called the focal distance of that point.*

LATUS RECTUM OF A PARABOLA *A chord of a parabola, passing through its focus and perpendicular to its axis, is called the latus rectum of the parabola.*

In the figure given below, LFL' is the latus rectum of the given parabola with focus F.

LENGTH OF LATUS RECTUM OF A PARABOLA

Let LFL' be the latus rectum of a parabola $y^2 = 4ax$ and let $|FL| = l$.

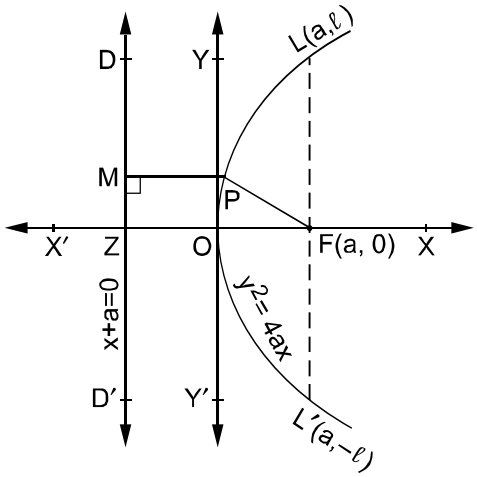
Then, clearly the coordinates of L and L' are $L(a, l)$ and $L'(a, -l)$.

Since the point $L(a, l)$ lies on the parabola $y^2 = 4ax$, we have

$$l^2 = 4a^2 \Rightarrow l = 2a.$$

$\therefore \quad |LL'| = 2l = 4a.$

Hence, the length of the latus rectum is $4a$.



The end points of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

The equation of the latus rectum is $x = a$.

SUMMARY

$y^2 = 4ax$, $a > 0$ is a parabola whose

(i) focus is $F(a, 0)$

(ii) vertex is $O(0, 0)$

(iii) directrix is the line $x + a = 0$

(iv) axis is the line $y = 0$

(v) length of the latus rectum is $4a$

(vi) latus rectum is the line $x = a$.

LEFT-HANDED PARABOLA (or 2nd Standard Form): $y^2 = -4ax$, $a > 0$

Proceeding as above, we can easily show that $y^2 = -4ax$ is the equation of a parabola whose focus is $F(-a, 0)$ and directrix is $x - a = 0$.

We summarise the properties of the parabola, $y^2 = -4ax$ as given below.

SUMMARY

$y^2 = -4ax$, $a > 0$ is a parabola whose

(i) focus is $F(-a, 0)$

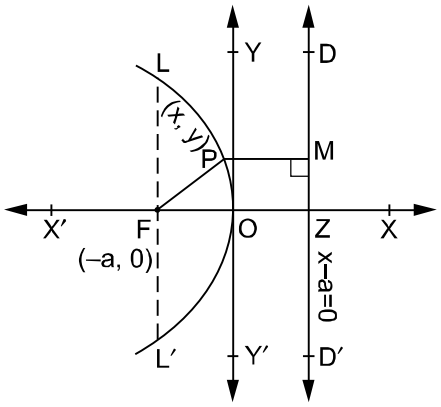
(ii) vertex is $O(0, 0)$

(iii) directrix is the line $x - a = 0$

(iv) axis is the line $y = 0$

(v) length of the latus rectum is $4a$

(vi) latus rectum is the line $x = -a$.



UPWARD PARABOLA (or 3rd Standard Form): $x^2 = 4ay$, $a > 0$

This is easy to show that $x^2 = 4ay$ is the equation of a parabola whose focus is $F(0, a)$ and directrix is $y + a = 0$.

We summarise the properties of the parabola, $x^2 = 4ay$ as given below.

SUMMARY

$x^2 = 4ay$, $a > 0$ is a parabola whose

(i) focus is $F(0, a)$

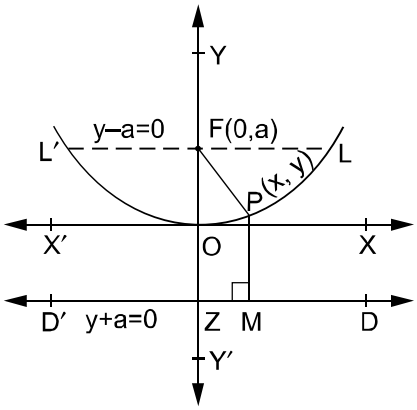
(ii) vertex is $O(0, 0)$

(iii) directrix is the line $y + a = 0$

(iv) axis is the line $x = 0$

(v) length of latus rectum is $4a$

(vi) latus rectum is the line $y - a = 0$.



DOWNWARD PARABOLA (or 4th Standard Form): $x^2 = -4ay$, $a > 0$

We can easily prove that $x^2 = -4ay$ is the equation of a parabola whose focus is $F(0, -a)$ and directrix is $y - a = 0$.

We summarise the properties of the parabola, $x^2 = -4ay$ as given below.

SUMMARY

$x^2 = -4ay$, $a > 0$ is a parabola whose

(i) focus is $F(0, -a)$

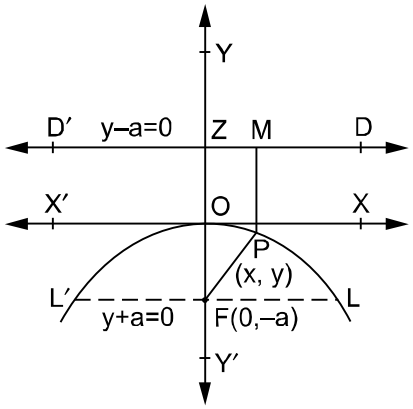
(ii) vertex is $O(0, 0)$

(iii) directrix is the line $y - a = 0$

(iv) axis is the line $x = 0$

(v) length of latus rectum is $4a$

(vi) latus rectum is the line $y + a = 0$.



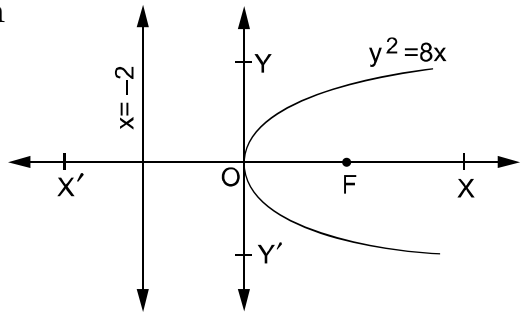
MAIN FACTS ABOUT ALL TYPES OF PARABOLAS

Parabola (Equation)	Focus	Vertex	Equation of directrix	Equation of axis	Length of latus rectum	Equation of latus rectum
(i) $y^2 = 4ax, a > 0$ (Right-handed)	$(a, 0)$	$(0, 0)$	$x + a = 0$	$y = 0$	$4a$	$x - a = 0$
(ii) $y^2 = -4ax, a > 0$ (Left-handed)	$(-a, 0)$	$(0, 0)$	$x - a = 0$	$y = 0$	$4a$	$x + a = 0$
(iii) $x^2 = 4ay, a > 0$ (Upward)	$(0, a)$	$(0, 0)$	$y + a = 0$	$x = 0$	$4a$	$y - a = 0$
(iv) $x^2 = -4ay, a > 0$ (Downward)	$(0, -a)$	$(0, 0)$	$y - a = 0$	$x = 0$	$4a$	$y + a = 0$

SOLVED EXAMPLES

EXAMPLE 1 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola* $y^2 = 8x$.

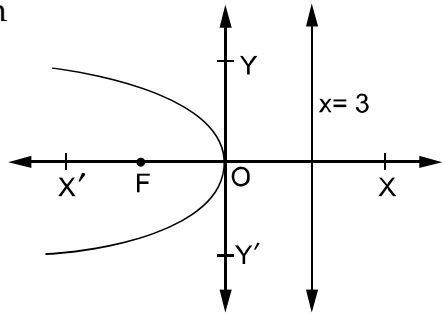
SOLUTION The given equation is of the form $y^2 = 4ax$, where $4a = 8$, i.e., $a = 2$.



This is a *right-handed parabola*.
Its focus is $F(a, 0)$, i.e., $F(2, 0)$.
Its vertex is $O(0, 0)$.
The equation of the directrix is $x = -a$, i.e., $x = -2$.
Its axis is x-axis, whose equation is $y = 0$.
Length of latus rectum $= 4a = (4 \times 2)$ units $= 8$ units.

EXAMPLE 2 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola* $y^2 = -12x$.

SOLUTION The given equation is of the form $y^2 = -4ax$, where $4a = 12$, i.e., $a = 3$.



This is a *left-handed parabola*.
Its focus is $F(-a, 0)$, i.e., $F(-3, 0)$.
Its vertex is $O(0, 0)$.
The equation of the directrix is $x = a$, i.e., $x = 3$.
Its axis is x-axis, whose equation is $y = 0$.
Length of latus rectum $= 4a = (4 \times 3)$ units $= 12$ units.

EXAMPLE 3 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola* $x^2 = 6y$.

SOLUTION The given equation is of the form $x^2 = 4ay$, where $4a = 6$, i.e., $a = \frac{3}{2}$.

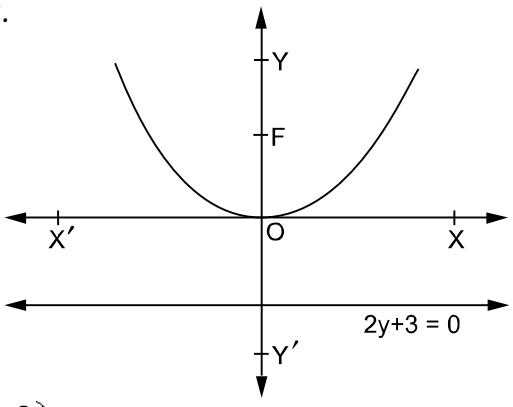
So, this is a case of *upward parabola.*

Its focus is $F(0, a)$, i.e., $F\left(0, \frac{3}{2}\right)$.

Its vertex is $O(0, 0)$.

The equation of the directrix is $y = -a$, i.e., $y = \frac{-3}{2}$, i.e., $2y + 3 = 0$.

Its axis is y-axis, whose equation is $x = 0$.



Length of latus rectum $= 4a = \left(4 \times \frac{3}{2}\right)$ units $= 6$ units.

EXAMPLE 4 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of latus rectum of the parabola* $x^2 = -16y$.

SOLUTION The given equation is of the form $x^2 = -4ay$, where $4a = 16$, i.e., $a = 4$.

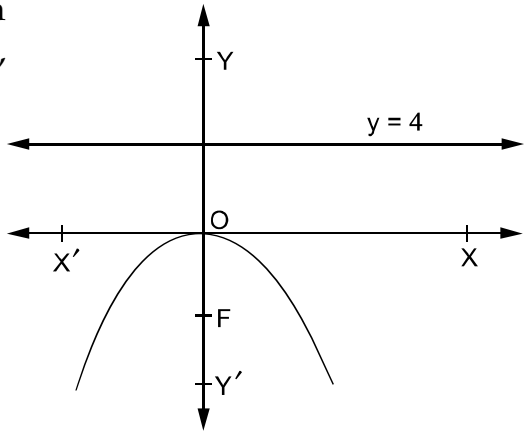
So, it is a case of *downward parabola*.

Its focus is $F(0, -a)$, i.e., $F(0, -4)$.

Its vertex is $O(0, 0)$.

The equation of the directrix is $y = a$, i.e., $y = 4$.

Its axis is y-axis, whose equation is $x = 0$.



Length of latus rectum $= 4a = (4 \times 4)$ units $= 16$ units.

EXAMPLE 5 *Find the equation of the parabola with focus at* $F(3, 0)$ *and directrix* $x = -3$.

SOLUTION Since the focus lies on the x-axis, so x-axis is the axis of the parabola.

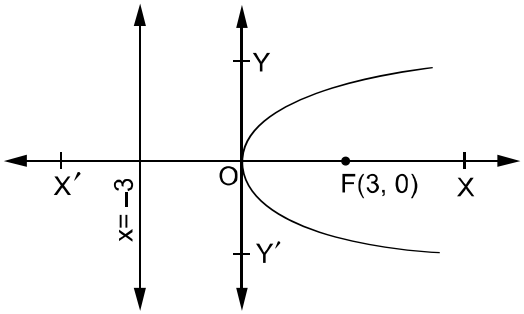
Focus $F(3, 0)$ lies to the right-hand side of the origin.

So, it is a *right-handed parabola.*

Let the required equation be $y^2 = 4ax$.

Then, focus is $F(a, 0)$. So, $a = 3$.

Hence, the required equation is $y^2 = 12x$.



EXAMPLE 6 *Find the equation of the parabola with vertex at the origin and $y + 3 = 0$ as its directrix. Also, find its focus.*

SOLUTION Let the vertex of the parabola be $O(0, 0)$.

Now, $y + 3 = 0 \Rightarrow y = -3$.

Thus, the directrix is a line parallel to the x-axis at a distance of 3 units below the x-axis.

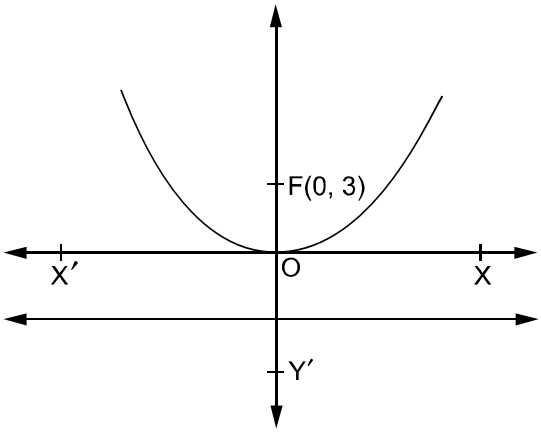
So, the focus of the parabola lies on the y-axis.

Consequently, the focus is $F(0, 3)$.

Hence, the equation of the parabola is

$$x^2 = 4ay, \quad \text{where } a = 3,$$

i.e., $x^2 = 12y$.



EXAMPLE 7 *Find the equation of the parabola with vertex at* $(0, 0)$ *and focus at* $(0, 2)$. *Also, find the equation of its directrix.*

SOLUTION Let $O(0, 0)$ be the vertex and $F(0, 2)$ be the focus of the given parabola.

Since focus lies on the y-axis, so y-axis is the axis of the parabola.

Also, the focus lies above the x-axis.

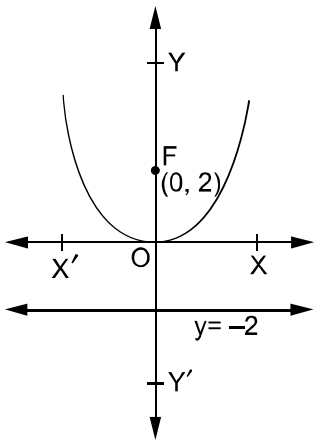
So, it is a case of *upward parabola.*

Let the equation of the parabola be $x^2 = 4ay$.

Then, its focus is $F(0, a)$ and so $a = 2$.

Hence, the required equation is $x^2 = 8y$.

Also, the equation of the directrix is $y = -2$.



EXAMPLE 8 *Find the equation of the parabola with vertex at the origin, passing through the point* $P(3, -4)$ *and symmetric about the y-axis.*

SOLUTION It is given that the vertex of the parabola is $O(0, 0)$ and it is symmetric about the y-axis.

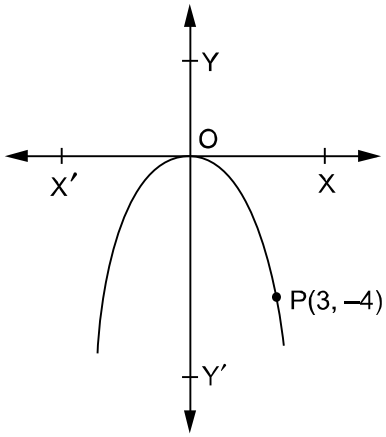
So, its equation is $x^2 = 4ay$ or $x^2 = -4ay$.

Since the parabola passes through the point $P(3, -4)$, so it lies in the 4th quadrant.

$\therefore$ it is a *downward parabola.*

Let its equation be $x^2 = -4ay$.

Since it passes through the point $P(3, -4)$, we have



$$3^2 = -4 \times a \times (-4) \Rightarrow a = \frac{9}{16}.$$

So, the required equation is

$$x^2 = -4 \times \frac{9}{16} y \Rightarrow x^2 = \frac{-9}{4} y \Rightarrow 4x^2 + 9y = 0.$$

EXAMPLE 9 *Find the equation of the parabola with vertex at the origin, the axis along the x-axis and passing through the point $P(2, 3)$.*

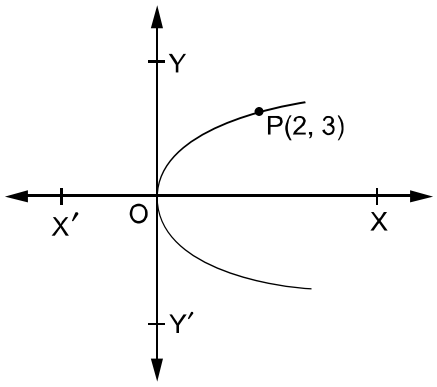
SOLUTION It is given that the vertex of the parabola is $O(0, 0)$ and its axis lies along the x-axis.

So, its equation is $y^2 = 4ax$ or $y^2 = -4ax$.

Since it passes through the point $P(2, 3)$, so it lies in the first quadrant.

$\therefore$ it is a *right-handed parabola.*

So, its equation is $y^2 = 4ax$.



Now, $P(2, 3)$ lies on it, so

$$3^2 = 4a \times 2 \Rightarrow a = \frac{9}{8}.$$

Hence, the required equation is $y^2 = 4 \times \frac{9}{8} x \Rightarrow y^2 = \frac{9}{2} x.$

EXERCISE 22

1. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:

(i) $y^2 = 12x$ (ii) $y^2 = 10x$ (iii) $3y^2 = 8x$

2. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:

(i) $y^2 = -8x$ (ii) $y^2 = -6x$ (iii) $5y^2 = -16x$

3. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:

(i) $x^2 = 16y$ (ii) $x^2 = 10y$ (iii) $3x^2 = 8y$

4. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:

(i) $x^2 = -8y$ (ii) $x^2 = -18y$ (iii) $3x^2 = -16y$

5. Find the equation of the parabola with vertex at the origin and focus at $F(-2, 0)$.

6. Find the equation of the parabola with focus $F(4, 0)$ and directrix $x = -4$.

7. Find the equation of the parabola with focus $F(0, -3)$ and directrix $y = 3$.

8. Find the equation of the parabola with vertex at the origin and focus $F(0, 5)$.

9. Find the equation of the parabola with vertex at the origin, passing through the point $P(5, 2)$ and symmetric with respect to the y-axis.

10. Find the equation of the parabola which is symmetric about the y-axis and passes through the point $P(2, -3)$.

ANSWERS (EXERCISE 22)

1. (i) $F(3, 0)$, $O(0, 0)$, $x + 3 = 0$, $y = 0$, 12 units

(ii) $F\left(\frac{5}{2}, 0\right)$, $O(0, 0)$, $2x + 5 = 0$, $y = 0$, 10 units

(iii) $F\left(\frac{2}{3}, 0\right)$, $O(0, 0)$, $3x + 2 = 0$, $y = 0$, $\frac{8}{3}$ units

2. (i) $F(-2, 0)$, $O(0, 0)$, $x = 2$, $y = 0$, 8 units

(ii) $F\left(\frac{-3}{2}, 0\right)$, $O(0, 0)$, $2x - 3 = 0$, $y = 0$, 6 units

(iii) $F\left(\frac{-4}{5}, 0\right)$, $O(0, 0)$, $5x - 4 = 0$, $y = 0$, $\frac{16}{5}$ units

3. (i) $F(0, 4)$, $O(0, 0)$, $y + 4 = 0$, $x = 0$, 16 units

(ii) $F\left(0, \frac{5}{2}\right)$, $O(0, 0)$, $2y + 5 = 0$, $x = 0$, 10 units

(iii) $F\left(0, \frac{2}{3}\right)$, $O(0, 0)$, $3y + 2 = 0$, $x = 0$, $\frac{8}{3}$ units

4. (i) $F(0, -2)$, $O(0, 0)$, $y = 2$, $x = 0$, 8 units

(ii) $F\left(0, \frac{-9}{2}\right)$, $O(0, 0)$, $2y - 9 = 0$, $x = 0$, 18 units

(iii) $F\left(0, \frac{-4}{3}\right)$, $O(0, 0)$, $3y - 4 = 0$, $x = 0$, $\frac{16}{3}$ units

5. $y^2 = -8x$ **6.** $y^2 = 16x$

7. $x^2 = -12y$ **8.** $x^2 = 20y$

9. $2x^2 = 25y$ **10.** $3x^2 = -4y$